On the decomposition of acoustic source radiation patterns measured with surrounding spherical microphone arrays

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Introduction

This article discusses the measurement of acoustic source radiation patterns with surrounding spherical microphone arrays in the context of acoustic holography. For practical reasons the spherical wave spectrum, which forms the basis for spherical holography, is determined through spherical harmonic transform (SHT) of discrete observations on the sphere. The discrete SHT and its limitations are addressed, particularly with emphasis on spherical harmonic order truncation and angular sampling; the existence and kind of spherical harmonic decompositions for different fundamental angular sampling schemes are summarized and compared for a varying number of sampling points. In conclusion, feasible array configurations for accurate acoustic holography are outlined with respect to numerical accuracy and hardware efficiency.

Spherical Wave Spectrum

The spherical harmonic transform (Fourier analysis) provides a decomposition of acoustic wave fields into their spherical wave components. The expansion coefficients are also referred to as spherical wave spectrum [1, 2]. The angular portions of the solution are typically combined into single functions, the spherical harmonics (SH)

$$Y_n^m(\theta,\phi) = \sqrt{\frac{(2n+1)}{4\pi}\frac{(n-m)!}{(n+m)!}}P_n^m(\cos\theta)e^{im\phi}, \qquad (1)$$

where n denotes the order and m the degree of the SH, P_n^m the associated Legendre functions, ϕ and θ are the azimuth and zenith angles, respectively, and $i=\sqrt{-1}$. The normalization in Eq. (1) is chosen such that

$$\int_{\mathbb{S}^2} Y_n^m(\theta,\phi)Y_{n'}^{m'}(\theta,\phi)^* d\Omega = \delta_{nn'}\delta_{mm'} \qquad (2)$$

where δ_{ij} is the Kronecker delta, $d\Omega = \sin\theta d\theta d\phi$ denotes the standard rotation invariant measure on the 2-sphere $\mathbb{S}^2$, and $(\cdot)^*$ denotes complex conjugation.

The complex spherical harmonic expansion coefficients ψ_{nm} can be determined from the sound pressure $p(r_0,\theta,\phi)$ on a sphere with radius $r=r_0$ using forward harmonic transform [3]

$$\psi_{nm}(r_0) = \int_{\mathbb{S}^2} p(r_0,\theta,\phi)Y_n^m(\theta,\phi)^* d\Omega, \qquad (3)$$

and the expansion in terms of SH becomes

$$p(r_0,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\psi_{nm}(r_0)Y_n^m(\theta,\phi). \qquad (4)$$

The spherical wave spectrum ψ_{nm} uniquely defines the radiated sound pressure and thus provides a reasonable description format for acoustic radiation analysis and synthesis.

Discrete Spherical Harmonic Transform

The spherical wave spectrum ψ_{nm} can be determined from a limited number of discrete observations on the sphere. This requires to suitably capture the sound field, considering the spatial sampling theorem and the free field assumption.

It can be assumed that real sound sources exhibit a band-limited spherical wave spectrum, which only consists of components with orders $n \le \mathrm{N}$. In this case, Eq. (4) can be reformulated in vector-matrix notation

$$\boldsymbol{p}_\mathrm{N} = \boldsymbol{Y}_\mathrm{N}\,\boldsymbol{\psi}_\mathrm{N}, \qquad (5)$$

where $\boldsymbol{Y}_\mathrm{N}$ consists of sampled spherical harmonics $\boldsymbol{y}_\mathrm{N}$, $\boldsymbol{\psi}_\mathrm{N}$ denotes the vector of corresponding SH coefficients, and $\boldsymbol{p}_\mathrm{N}$ the sound pressure at the discrete observation points. Determining $\boldsymbol{\psi}_\mathrm{N}$ from $\boldsymbol{p}_\mathrm{N}$ requires to invert $\boldsymbol{Y}_\mathrm{N}$, which in many cases is badly conditioned.

The singular value decomposition (SVD) provides a well established solution method [4]. It decomposes the matrix $\boldsymbol{Y}_\mathrm{N} = \boldsymbol{U}\ \boldsymbol{S}\ \boldsymbol{V}^\mathrm{T}$ into a diagonal matrix $\boldsymbol{S}$ and two orthogonal matrices $\boldsymbol{U}$ and $\boldsymbol{V}$, containing the singular values and singular vectors of $\boldsymbol{Y}_\mathrm{N}$, respectively. Keeping only the K non-vanishing singular values in $\tilde{\boldsymbol{S}}$ and cropping the orthogonal matrices accordingly regularizes the inverse $\boldsymbol{Y}_\mathrm{N}^\dagger = \tilde{\boldsymbol{V}}\ \tilde{\boldsymbol{S}}^{-1}\ \tilde{\boldsymbol{U}}^\mathrm{T}$. The solution of Eq. (5) becomes

$$\tilde{\boldsymbol{\psi}}_\mathrm{N} = \boldsymbol{Y}_\mathrm{N}^\dagger\,\boldsymbol{p}. \qquad (6)$$

Depending on the dimensions of $\boldsymbol{Y}_\mathrm{N}$, the number of non-vanishing singular values, $\mathrm{K} = \#\{s_l : s_l \ge a\,s_\mathrm{max}\}$, with $0 < a < 1$, and the number of observation points, L, Eq. (6) has the following properties:

1. $\mathrm{K} = (\mathrm{N}+1)^2 \le \mathrm{L}$: *Discrete spherical harmonic "transform" (DSHT)* or spectral analysis,
2. $\mathrm{K} = \mathrm{L} \le (\mathrm{N}+1)^2$: *Discrete spherical harmonic interpolation (DSHI)* or pseudo-spectral analysis,

3. $\mathrm{K} < \min[(\mathrm{N}+1)^2, \mathrm{L}]$: *Discrete spherical harmonic approximation (DSHA).*

In the DSHT case, the pseudo-inverse provides an exact SH analysis $\tilde{\boldsymbol{\psi}}_{\mathrm{N}} = \boldsymbol{\psi}_{\mathrm{N}}$ for a strictly band-limited radiation pattern $\boldsymbol{p} = \boldsymbol{p}_{\mathrm{N}}$; the inverse becomes $(\boldsymbol{Y}_{\mathrm{N}}^{\mathrm{T}}\boldsymbol{Y}_{\mathrm{N}})^{-1}\boldsymbol{Y}_{\mathrm{N}}^{\mathrm{T}}$ and inverts Eq. (5) from the left. In general, exact DSHT is required to provide numerically stable computations for acoustic holography. In the DSHI case, the pseudo-inverse behaves as interpolation that achieves an exact representation at the sampling nodes and approximates arbitrary radiation patterns $\boldsymbol{p}_{\mathrm{N}} = \boldsymbol{p}$ by order-limited spherical harmonics expansion; the inverse becomes $\boldsymbol{Y}_{\mathrm{N}}^{\mathrm{T}}(\boldsymbol{Y}_{\mathrm{N}}\boldsymbol{Y}_{\mathrm{N}}^{\mathrm{T}})^{-1}$ and inverts Eq. (5) from the right. For the DSHA case, neither exact transform, nor exact interpolation is feasible.

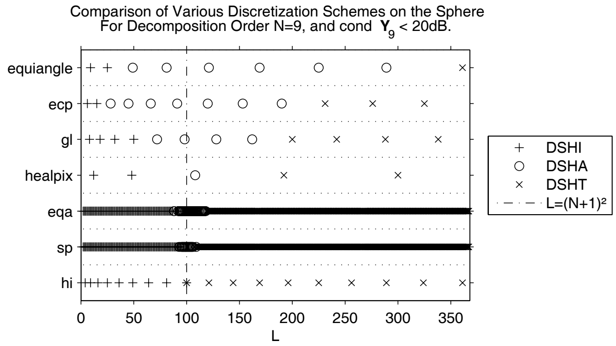
Fig. 1 compares fundamental discretization schemes with a varying number of sampling points L on the sphere. It provides a classification into the above mentioned three part scheme for SH order truncation $\mathrm{N} = 9$. The following spherical discretization schemes are considered: Extremal points for hyperinterpolation (*hi*), spiral points (*sp*), equal-area partitions (*eqa*), HEALPix (*healpix*), Gauss-Legendre grid (*gl*), equi-distant cylindrical grid (*ecp*), and equi-angle grid (*equiangle*), cf. [6].

The condition number of the matrix $\boldsymbol{Y}$, *i.e.* the ratio between its maximum and minimum singular value, measures the sensibility and stability of the solution of the inverse problem. If the condition number is too large, regularization by SVD is required. In Fig. 1, the regularization parameter was chosen to be $a = -20$dB and matrices with condition numbers $\mathrm{cond}\{\boldsymbol{Y}_9\} \leq 20$dB were left or right inverted without regularization, *i.e.* the case of DSHT or DSHI. Greater condition numbers are considered to require limitation by regularization, hence become a DSHA.

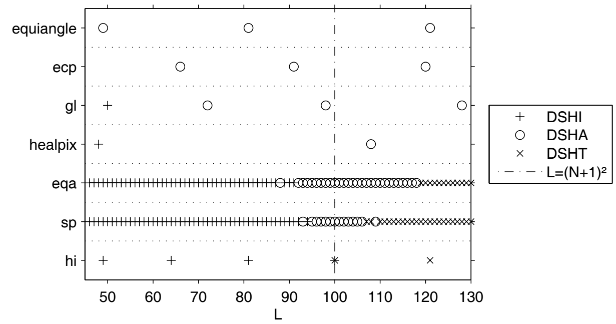
It can be seen from Fig. 1(b) that near the critical number of sampling nodes, $\mathrm{L} = (\mathrm{N}+1)^2$, most discretization schemes only allow for DSHA. In the context of critical sampling, the extremal points designed for *hyperinterpolation* provide an exact and well-conditioned inverse, which is both left and right inverse to the spherical harmonics expansion. This inversion is then called *hyperinterpolation* and provides both DSHT and DSHI.

Conclusions

In this article we have discussed the spherical harmonic transform of acoustic source radiation patterns measured at discrete observation points on a surrounding sphere. The inverse of the matrix $\boldsymbol{Y}_N$ is regularized using SVD. As a result, a classification of different fundamental discretization schemes with respect to the quality of acoustic holography is given. The analysis of simulation results clearly shows that the *hyperinterpolation*, *spiral points*, and *equal area* partitioning schemes provide an exact transform for a minimal number of required sampling nodes and thus reduce the measurement effort.



(a)



(b)

Figure 1: Existence and kind of SH decompositions for different spherical discretization schemes with L sampling points and order truncation $\mathrm{N} = 9$; (b) highlights the region around the critical number of sampling points (vertical dash-dot line) to illustrate the unique behavior of the *hyperinterpolation*.

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